Hamiltonians for curves

2002 J. Phys. A: Math. Gen. 35 6571

(http://iopscience.iop.org/0305-4470/35/31/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6571–6587 PII: S0305-4470(02)35170-9

Hamiltonians for curves

R Capovilla[1], C Chryssomalakos[2] and J Guven[2]

[1] Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apdo Postal 14-740, 07000 México DF, Mexico
[2] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apdo Postal 70-543, 04510 México DF, Mexico

Received 25 March 2002
Published 26 July 2002
Online at stacks.iop.org/JPhysA/35/6571

Abstract
We examine the equilibrium conditions of a curve in space when a local energy penalty is associated with its extrinsic geometrical state characterized by its curvature and torsion. To do this we tailor the theory of deformations to the Frenet–Serret frame of the curve. The Euler–Lagrange equations describing equilibrium are obtained; Noether's theorem is exploited to identify the constants of integration of these equations as the Casimirs of the Euclidean group in three dimensions. While this system appears not to be integrable in general, it *is* in various limits of interest. Let the energy density be given as some function of the curvature and torsion, $f(\kappa, \tau)$. If f is a linear function of either of its arguments but otherwise arbitrary, we claim that the first integral associated with rotational invariance permits the torsion τ to be expressed as the solution of an algebraic equation in terms of the bending curvature, κ. The first integral associated with translational invariance can then be cast as a quadrature for κ or for τ.

PACS numbers: 02.30.Xx, 11.10.Ef, 61.41.+e

1. Introduction

Consider a curve in space. Suppose that the curve is sufficiently smooth so that the Frenet–Serret frame adapted to it is defined. The curvature κ and the torsion τ then provide a complete characterization of the curve; once they are known, it can be reconstructed up to Euclidean motions. In this paper we examine local reparametrization invariant Hamiltonians for curves of the form

$$H = \int \mathrm{d}s\, f(\kappa, \tau) \tag{1}$$

where s denotes the arclength, and f is any scalar under reparametrizations.

Such Hamiltonians play a role both in the static and in the kinematic description of curves. In the former, we interpret the Hamiltonian as the energy of the physical system; in particular, an energy of the form $f = \kappa^2$ penalizing bending models the stiffness of a polymer [1, 2], and it has been used to model the elastic properties of DNA (see, e.g., [3, 4]). When the energy depends only on the curvature, as Bernoulli and Euler both knew, the equilibrium conditions are integrable [5]: the torsion is always a function of κ, and κ satisfies a quadrature. Our focus will be on the general case (1). While a linear dependence on τ, associated with a constraint on the total torsion, has been considered [6], very little appears to be known beyond that [7]. At the very least, one is interested in a quadratic dependence on both κ and τ, the second-order terms in a Taylor expansion of f. Such terms appear in the Hamiltonian describing chiral polymers [8].

An additional motivation for studying these Hamiltonians is the role they play in the connection between the motion of curves and integrable systems: the equation describing the evolution of some function of the curvature and the torsion with respect to certain length-preserving vector fields coincides with the nonlinear Schrödinger equation; other functions give other known integrable equations. These curve motions appear in a number of contexts: vortex filaments and patches in fluids [9–11], classical magnetic spin chains [12, 13], interface dynamics [14], etc. Specific Hamiltonians of the form (1) emerge as conserved quantities under these motions [15, 16].

Our strategy in the study of the Hamiltonians (1) will be to exploit Noether's theorem to identify the equilibrium conditions as conservation laws associated with the Euclidean invariance of the energy. These conservation laws, in turn, permit us to identify first integrals of the equilibrium conditions. Remarkably, in certain cases, these integrals can be combined to provide a quadrature for either κ or τ. The constants of integration are the Casimir invariants of the Euclidean group. We show that, in addition to the well-known case of a pure bending Hamiltonian, a pure torsion Hamiltonian also leads to integrable equilibrium conditions. This is surprising because, in contrast to the curvature which depends on two derivatives of the embedding function for the curve, the torsion depends on three. The Euler–Lagrange equations which result involve six derivatives; as such one would not expect them to be tractable. The torsion is determined by a quadrature. For a polynomial f, the potential appearing in this quadrature is a rational function. We identify other Hamiltonians with a joint dependence on κ and τ which are reducible to a quadrature. In general, unfortunately, it does not appear to be possible to identify a quadrature. The integrals of the motion can, however, be used to reduce the equilibrium conditions to the motion of a fictitious particle in two dimensions. In any case, this reduction should be helpful for studying these systems.

For most physically realistic materials the local arclength will be constant. This is because there will be a large energy penalty associated with stretching the curve. Suppose that an arbitrary deformation is decomposed into tangential and normal parts. The constraint on the arclength can then be phrased in terms of the corresponding response of the tangential deformation to its normal counterpart. However, as we will discuss below, a tangential deformation is a reparametrization so that the corresponding change in the Hamiltonian can always be absorbed in a divergence; as such, it cannot affect the Euler–Lagrange equations. Thus, whether or not we decide to implement a constraint on arclength, the equations themselves describing equilibrium are unchanged.

The paper is organized as follows. In section 2, we begin by giving a self-contained account of the theory of deformations of a curve tailored to the Frenet–Serret frame. In distinction to earlier work, it is not necessary to implement the constraint associated with the locally arclength preserving deformations. We obtain directly simple expressions for the variation of the curvature and torsion. In section 3, we analyse the consequences of the invariance of the Hamiltonian under reparametrizations as well as under Euclidean motions. We show how to obtain expressions for the internal forces and torques on any segment of a curve and their relationship with the equilibrium conditions. Systems which depend at most on the curvature κ are the subject of section 4. This is extended to systems that depend on the torsion τ in section 5. In section 6, systems that depend on both curvature and torsion are considered. In section 7 we briefly consider perturbations of the equilibrium conditions. Section 8 relates some of the results of this paper to recursion schemes that appear in the kinematics of curves.

2. Curve deformations

In this section, we describe the geometry of embedded curves in three-dimensional space in terms of the Frenet–Serret basis for the curve, and the effect of a small deformation of the curve on its geometry.

Consider a curve in space described by the embedding $\boldsymbol{x} = \boldsymbol{X}(s)$, where $\boldsymbol{X} = (X^1, X^2, X^3)$. The unit tangent to the curve is given by $\boldsymbol{t} = \boldsymbol{X}'$, where the prime denotes a derivative with respect to the arclength s. Clearly, the 'acceleration' $\boldsymbol{t}'$ is orthogonal to $\boldsymbol{t}$. However, $\boldsymbol{t}''$ is not. The classical Frenet–Serret equations

$$\boldsymbol{t}' = \kappa \boldsymbol{n}_1 \qquad \boldsymbol{n}_1' = -\kappa \boldsymbol{t} + \tau \boldsymbol{n}_2 \qquad \boldsymbol{n}_2' = -\tau \boldsymbol{n}_1 \tag{2}$$

describe the construction of an orthonormal basis $\{\boldsymbol{t}, \boldsymbol{n}_1, \boldsymbol{n}_2\}$ along the curve. We choose an orientation with $\boldsymbol{n}_2 = \boldsymbol{t} \times \boldsymbol{n}_1$. κ and τ are, respectively, the geodesic curvature and torsion. The fundamental theorem for curves tells us that the Frenet–Serret curvatures κ and τ determine the curve up to rigid motions [17]. The actual curve can always be reconstructed from its curvatures. Thus, they provide a natural set of auxiliary variables. Any local geometrical scalar defined along the curve can in principle always be expressed as a function of the curvatures and their derivatives.

We now analyse the change in the geometry of the curve due to an infinitesimal deformation of its embedding in space, $\boldsymbol{X}(s) \to \boldsymbol{X}(s) + \delta \boldsymbol{X}(s)$. Let us first decompose the deformation into its tangential and normal parts with respect to the basis $\{\boldsymbol{t}, \boldsymbol{n}_1, \boldsymbol{n}_2\}$,

$$\delta \boldsymbol{X} = \Psi_{\|} \boldsymbol{t} + \Psi_1 \boldsymbol{n}_1 + \Psi_2 \boldsymbol{n}_2. \tag{3}$$

This is a convenient strategy when one is interested in the variation of reparametrization-invariant geometrical quantities. Tangential deformations are reparametrizations of the curve. We will use the following two facts: the tangential deformations of any scalar f and the infinitesimal arclength are given by

$$\delta_{\|} f = \Psi_{\|} f' \qquad \delta_{\|}\, \mathrm{d}s = \mathrm{d}s\, \Psi_{\|}'. \tag{4}$$

Now consider the normal part of the deformation. A normal deformation of $\mathrm{d}s$ is

$$\delta_\perp\, \mathrm{d}s = -\mathrm{d}s\, \kappa \Psi_1. \tag{5}$$

This result implies that, for any scalar f,

$$\delta_\perp (f') = \kappa f' \Psi_1 + (\delta_\perp f)'. \tag{6}$$

In particular, for the three scalar functions $\boldsymbol{X}$, this implies

$$\delta_\perp \boldsymbol{t} = \kappa \Psi_1 \boldsymbol{t} + (\Psi_1 \boldsymbol{n}_1 + \Psi_2 \boldsymbol{n}_2)'. \tag{7}$$

We now use the Frenet–Serret equations to cast $\delta_\perp \boldsymbol{t}$ as a normal vector (a unit vector and its variation are orthogonal):

$$\delta_\perp \boldsymbol{t} = (\Psi_1' - \tau \Psi_2) \boldsymbol{n}_1 + (\Psi_2' + \tau \Psi_1) \boldsymbol{n}_2. \tag{8}$$

Similarly we have

$$\delta_\perp \boldsymbol{t}' = \kappa \Psi_1 \boldsymbol{t}' + (\delta_\perp \boldsymbol{t})' \tag{9}$$

using the Frenet–Serret equations and (8) we obtain

$$\delta_\perp \boldsymbol{t}' = -\kappa(\Psi_1' - \tau\Psi_2)\boldsymbol{t} + [\Psi_1'' + (\kappa^2 - \tau^2)\Psi_1 - 2\tau\Psi_2' - \tau'\Psi_2]\,\boldsymbol{n}_1 + (2\tau\Psi_1' + \tau'\Psi_1 + \Psi_2'' - \tau^2\Psi_2)\,\boldsymbol{n}_2. \quad (10)$$

We are now in a position to evaluate the normal variations of the two Frenet–Serret curvatures. To evaluate $\delta_\perp\kappa$, we take a variation of the first of (2): $\delta_\perp \boldsymbol{t}' = (\delta_\perp\kappa)\boldsymbol{n}_1 + \kappa\delta_\perp\boldsymbol{n}_1$. Dotting with $\boldsymbol{n}_1$ we obtain $\delta_\perp\kappa = \boldsymbol{n}_1 \cdot \delta_\perp\boldsymbol{t}'$. From (10) we read off that

$$\delta_\perp\kappa = \Psi_1'' + (\kappa^2 - \tau^2)\Psi_1 - 2\tau\Psi_2' - \tau'\Psi_2. \quad (11)$$

For a planar curve, $\tau = 0$ and $\delta_\perp\kappa = \Psi_1'' + \kappa^2\Psi_1$. Deformations lifting the curve off the plane do not affect the value of κ to first order in the deformation.

To evaluate $\delta_\perp\tau$, we take a variation of the FS equation for $\boldsymbol{n}_1'$ and dot with $\boldsymbol{n}_2$. We have

$$\delta_\perp\tau = \kappa\boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{t} + \boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{n}_1'. \quad (12)$$

We now rewrite the second term on the right-hand side as

$$\begin{aligned}\boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{n}_1' &= \kappa\Psi_1(\boldsymbol{n}_2 \cdot \boldsymbol{n}_1') + \boldsymbol{n}_2 \cdot (\delta_\perp\boldsymbol{n}_1)' \\ &= \kappa\tau\Psi_1 + (\boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{n}_1)' \\ &= \kappa\tau\Psi_1 + \left[\frac{1}{\kappa}\boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{t}'\right]' \end{aligned} \quad (13)$$

where we have applied (6) to $\boldsymbol{n}_1'$, and used the FS equations for both $\boldsymbol{n}_1'$ and $\boldsymbol{n}_2'$. Substituting for $\boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{t}$ and $\boldsymbol{n}_2 \cdot \delta_\perp\boldsymbol{t}'$, we obtain

$$\delta_\perp\tau = \kappa(\Psi_2' + 2\tau\Psi_1) + \left\{\frac{1}{\kappa}[2\tau\Psi_1' + \tau'\Psi_1 + \Psi_2'' - \tau^2\Psi_2]\right\}'. \quad (14)$$

For an initially planar curve, $\delta_\perp\tau = \kappa\Psi_2' + (\Psi_2''/\kappa)'$. Only the deformation along the direction normal to the plane contributes. Suppose that $\delta_\perp\tau = 0$, then the equation $\kappa\Psi_2' + (\Psi_2''/\kappa)' = 0$ should not admit any solutions other than those which correspond to an Euclidean motion of the planar curve. To show this we note that, for a planar curve, $\kappa = \Theta'$, where Θ is the angle which the tangent makes with, say, the x-axis. Then this equation can be recast as $\partial_\Theta^2\Psi_2' + \Psi_2' = 0$, with independent solutions, $\Psi_2 = \sin\Theta$, $\cos\Theta$ and Ψ_2 a constant—a rotation about x, y and a translation. If $\delta_\perp\tau$ is constant, the only solution is the helix $\Psi_2 \propto \Theta$ generated by the planar curve.

For completeness, let us note that the normals vary according to

$$\delta_\perp\boldsymbol{n}_1 = -(\Psi_1' - \tau\Psi_2)\boldsymbol{t} + \frac{1}{\kappa}(\Psi_2'' - \tau^2\Psi_2 + 2\tau\Psi_1' + \tau'\Psi_1)\boldsymbol{n}_2 \quad (15)$$

$$\delta_\perp\boldsymbol{n}_2 = -(\Psi_2' + \tau\Psi_1)\boldsymbol{t} - \frac{1}{\kappa}(\Psi_2'' - \tau^2\Psi_2 + 2\tau\Psi_1' + \tau'\Psi_1)\boldsymbol{n}_1. \quad (16)$$

So far we have considered arbitrary deformations of the curve. There are special deformations that will be of interest in the following. In particular, a deformation $\boldsymbol{Y} = \delta\boldsymbol{X}$ preserves locally the arclength if $\delta_{\boldsymbol{Y}}\,\mathrm{d}s = 0$. In terms of the components this translates to

$$\boldsymbol{t} \cdot \boldsymbol{Y}' = Y_\parallel' - \kappa Y_1 = 0 \quad (17)$$

which implies that there exists a (non-unique) vector $\boldsymbol{Z}$ such that

$$\boldsymbol{t} \times \boldsymbol{Z} = \boldsymbol{Y}'. \quad (18)$$

This is the starting point of the filament model recursion scheme [16], where a family of locally arclength preserving vector fields $\{\boldsymbol{Y}_{(n)}\}$ is defined by $\boldsymbol{t} \times \boldsymbol{Y}_{(n)} = \boldsymbol{Y}'_{(n-1)}$, with $\boldsymbol{Y}_{(0)} = -\boldsymbol{t}$. We will have something to say about this in section 8.

Let us also note that if $(\delta_\perp\boldsymbol{X})' \cdot \boldsymbol{n}_i = 0$, or in components, $\Psi_1' - \tau\Psi_2 = 0$ and $\Psi_2' + \tau\Psi_1 = 0$, then $\delta_\perp\boldsymbol{n}_i = 0$ and $\delta_\perp\boldsymbol{t} = 0$; the Frenet–Serret basis is left unchanged by this type of deformation.

3. Invariance and symmetry

The Hamiltonian H for the curve depends locally on the geometry, and it possesses various symmetries, both local and global. The local symmetry is reparametrization invariant, and it restricts severely the form of H. The global symmetries are Euclidean motions: translations and rotations. They give rise to conservation laws.

3.1. Reparametrization invariance

The Hamiltonian H is, in general, a sum of terms, $H = H_1 + H_2 + \cdots$, each of which is invariant under reparametrizations of the curve. This results in the form

$$H = \int \mathrm{d}s\, f(\kappa, \tau, \kappa', \tau', \ldots) \tag{19}$$

where f is a scalar under reparametrizations, constructed out of the geometrical quantities that characterize the curve: the two curvatures and their derivatives. The lowest order (in derivatives of the embedding functions) non-trivial geometrical model depends only on the scalar κ. A simple model that penalizes bending of the curve is $f_1 = \boldsymbol{t}' \cdot \boldsymbol{t}' = \kappa^2$. At the next order, a dependence on τ as well as κ' is admitted. A term in f of the form τ^2 penalizes the torsion of the curve; one of the form $(\kappa')^2$ is a higher-order differential bending energy and will not be considered. We note that the natural Hamiltonians $f_2 \equiv \boldsymbol{n}_1' \cdot \boldsymbol{n}_1'$ and $f_3 \equiv \boldsymbol{n}_2' \cdot \boldsymbol{n}_2'$ are given by $f_2 = \kappa^2 + \tau^2$ and τ^2. While f_1 has been considered in considerable detail [18, 19], neither f_2 nor f_3 appears to have been considered.

The assumption that f is a scalar under reparametrizations implies that the first variation of the energy can always be written as

$$\delta H = \int \mathrm{d}s\, \mathcal{E}_i \Psi_i + \int \mathrm{d}s\, \mathcal{Q}' \tag{20}$$

where $\mathcal{E}_i$ denotes the normal projection of the Euler–Lagrange derivative of f, and $\mathcal{Q}$ is the Noether charge $(i, j, = 1, 2)$. The specific form of the first term follows from the fact that the tangential variation contributes only in a divergence. Indeed, using (4), the tangential part of the variation of the energy is always a total derivative

$$\delta_\| H = \int \mathrm{d}s (f \Psi_\|)'. \tag{21}$$

This implies that the Noether charge $\mathcal{Q}$, as a linear differential operator which operates on the deformation $\delta \boldsymbol{X}$, is of the form

$$\mathcal{Q} = f \Psi_\| + \mathcal{Q}_{(0)}{}^i \Psi_i + \mathcal{Q}_{(1)}{}^i \Psi_i' + \cdots \tag{22}$$

where, to construct the $\mathcal{Q}_{(n)}{}^i$, we use integration by parts to collect in a total derivative the normal deformations Ψ_i and their derivatives.

3.2. Translational invariance

The Hamiltonian H is also invariant under the rigid Euclidean motions: translations and rotations. Noether's theorem can then be exploited to determine the conditions of static equilibrium. Under an infinitesimal constant translation, $\delta \boldsymbol{X} = \boldsymbol{e}$, the energy associated with H stored on the segment AB (labelled by its endpoints) changes by an amount

$$\delta H_{AB} = \boldsymbol{e} \cdot \int_A^B \mathrm{d}s [\mathcal{E}^i \boldsymbol{n}_i - \boldsymbol{F}']. \tag{23}$$

Here we introduce the spatial vector $\boldsymbol{F}$ with

$$\mathcal{Q} = -\boldsymbol{e} \cdot \boldsymbol{F} \tag{24}$$

and it is constructed by specializing equation (22) to the case of a constant deformation.

With no external forces acting so that $\mathcal{E}_i = 0$, we may identify $\boldsymbol{F}$ as the internal force pulling or pushing at a point on the curve segment. (With our conventions, $\boldsymbol{F}$ is the force from the part with lower to the one with higher value of s.) In general, this force will not be tangential. Translational invariance implies $\delta H_{AB} = 0$. Because the endpoints A and B are arbitrary we deduce the local balance of forces

$$\mathcal{E}^i \boldsymbol{n}_i - \boldsymbol{F}' = 0. \tag{25}$$

In equilibrium, this implies the conservation law, $\boldsymbol{F}' = 0$, i.e. $\boldsymbol{F}$ is a constant vector along the curve. We thus associate a spatial vector with each curve, be it open or closed. The squared magnitude of this vector, F^2, is the first Casimir of the Euclidean group. The direction along which $\boldsymbol{F}$ points is often indicated by the symmetry of the configuration.

We have come up with three equations of static equilibrium, whereas we only possess two independent Euler–Lagrange equations. One of the former must therefore be a kinematical statement, or Bianchi identity, associated with the reparametrization invariance of H. Let us examine the three independent projections of the equilibrium conditions (25): we decompose $\boldsymbol{F}$ into parts tangential and normal to the curve,

$$\boldsymbol{F} = F_{\|}\boldsymbol{t} + F_1\boldsymbol{n}_1 + F_2\boldsymbol{n}_2. \tag{26}$$

Then equation (25) is equivalent to the three equations

$$F_{\|}' - \kappa F_1 = 0 \qquad F_1' + \kappa F_{\|} - \tau F_2 = \mathcal{E}_1 \qquad F_2' + \tau F_1 = \mathcal{E}_2. \tag{27}$$

Comparison with (17) shows that $\boldsymbol{F}$, seen as a deformation of the curve, preserves locally arclength, as expected. The first condition is independent of the Euler–Lagrange equations. It is the promised Bianchi identity associated with the reparametrization invariance of H. Note that these equations could also have been derived directly by considering the balance of forces, as in [20].

There is a non-trivial integrability condition on closed curves associated with the conservation law (25). Taking its projection onto $\boldsymbol{X}$ in equilibrium, we can immediately deduce that

$$\oint \mathrm{d}s\, F_{\|} = 0 \tag{28}$$

on any closed loop. We will comment on its geometrical origin in section 3.4.

3.3. Rotational invariance

Under an infinitesimal rotation $\delta\boldsymbol{X} = \boldsymbol{\Omega} \times \boldsymbol{X}$, we have that the energy H on a segment AB of the curve changes by

$$\delta H_{AB} = \boldsymbol{\Omega} \cdot \int_A^B \mathrm{d}s [\mathcal{E}^i \boldsymbol{n}_i \times \boldsymbol{X} - \boldsymbol{M}'] \tag{29}$$

where the spatial vector $\boldsymbol{M}$ is defined by

$$\mathcal{Q} = -\boldsymbol{\Omega} \cdot \boldsymbol{M} \tag{30}$$

and it is obtained from equation (22) by specializing it to the case of a constant rotation. We identify $\boldsymbol{M}$ as the torque with respect to the origin acting at a point on the curve segment. Rotational invariance of H_{AB} implies

$$\boldsymbol{M}' = \mathcal{E}^i \boldsymbol{n}_i \times \boldsymbol{X}. \tag{31}$$

We decompose $\boldsymbol{M}$ into the sum of the couple of $\boldsymbol{F}$ about the origin, plus an intrinsic, translationally invariant part,

$$\boldsymbol{M} = \boldsymbol{X} \times \boldsymbol{F} + \boldsymbol{T}. \tag{32}$$

Then the differential torque $\boldsymbol{T}$ satisfies

$$\boldsymbol{T}' = \boldsymbol{F} \times \boldsymbol{t}. \tag{33}$$

We emphasize that this equation does not depend on whether the curve is in equilibrium or not. While $\boldsymbol{M}$ is conserved in equilibrium, neither $\boldsymbol{X} \times \boldsymbol{F}$ nor $\boldsymbol{T}$ is. It is clear, however, that the projection of $\boldsymbol{T}$ onto $\boldsymbol{F}$, the second Casimir of the Euclidean group, is conserved

$$J = \boldsymbol{T} \cdot \hat{\boldsymbol{F}} \tag{34}$$

where $\hat{\boldsymbol{F}} = \boldsymbol{F}/F$, and for future convenience, we choose to fold $|\boldsymbol{F}|$ into the definition.

As we did earlier for $\boldsymbol{F}$, we can also decompose $\boldsymbol{T}$ into parts tangential and normal to the curve,

$$\boldsymbol{T} = T_{\|}\boldsymbol{t} + T_1\boldsymbol{n}_1 + T_2\boldsymbol{n}_2. \tag{35}$$

Then equation (33) is equivalent to

$$T_{\|}' - \kappa T_1 = 0 \qquad T_1' - \tau T_2 + \kappa T_{\|} = F_2 \qquad T_2' + \tau T_1 = -F_1 \tag{36}$$

where we use the convention $\boldsymbol{t} \cdot \boldsymbol{n}_1 \times \boldsymbol{n}_2 = 1$. The first equation, as was the case for $\boldsymbol{F}$ in equation (28), is also valid off-shell, and it says that $\boldsymbol{T}$, seen as a deformation of the curve preserves locally arclength.

If a curve is deformed along $\boldsymbol{T}$, i.e. $\delta\boldsymbol{X} = \boldsymbol{T}$, we find that the corresponding variations of the curvature and torsion satisfy

$$\delta_{\boldsymbol{T}}\kappa = \mathcal{E}_2 \qquad \delta_{\boldsymbol{T}}\tau = -\left(\frac{\mathcal{E}_1}{\kappa}\right)'. \tag{37}$$

Therefore, in equilibrium, not only deformations along $\boldsymbol{F}$, $\boldsymbol{M}$ correspond to rigid motions which leave the geometry unchanged, as expected, but also deformations along $\boldsymbol{T}$.

3.4. Adapted cylindrical coordinates

The two conserved vectors $\boldsymbol{F}$ and $\boldsymbol{M}$ together single out a cylindrical polar coordinate system $\{\rho, \theta, z\}$ [18, 21, 22]. Suppose $\boldsymbol{F} \neq 0$. We align our coordinate system such that $\hat{\boldsymbol{F}}$ points along the positive z-direction. We next perform a translation orthogonal to $\boldsymbol{F}$ so that $\hat{\boldsymbol{M}}$ is rotated into $\hat{\boldsymbol{F}}$. Then we have that

$$\boldsymbol{T} = J\hat{\boldsymbol{F}} + \boldsymbol{F} \times \boldsymbol{X}. \tag{38}$$

Alternatively, we can arrive at this expression by integrating equation (33),

$$\int \mathrm{d}s\, \boldsymbol{T}' = \int \mathrm{d}s (\boldsymbol{F} \times \boldsymbol{X})' - \int \mathrm{d}s\, \boldsymbol{F}' \times \boldsymbol{t} \tag{39}$$

and noting that at equilibrium the second term vanishes, so that $\boldsymbol{T}$ and $\boldsymbol{F} \times \boldsymbol{X}$ differ by a constant vector. Then contraction with $\hat{\boldsymbol{F}}$ reproduces equation (38), up to a constant translation.

The modulus

$$T^2 = J^2 + \rho^2 F^2 \tag{40}$$

determines the cylindrical radius in the adapted system in terms of T^2 and the two Casimirs. Typically, T^2 will be some function of κ and τ and their derivatives. To complete the construction, we describe the tangent vector in these coordinates, $\boldsymbol{t} = (\rho', \theta', z')$. Thus, the projection $\boldsymbol{t} \cdot \hat{\boldsymbol{F}}$ determines z as a quadrature

$$z' = \frac{F_\parallel}{F}. \tag{41}$$

This provides the promised geometrical interpretation of the global conservation law (28). Similarly, the projection $\boldsymbol{t} \cdot \hat{\boldsymbol{T}}$ determines θ as a quadrature,

$$F\rho^2\theta' = Jz' - T_\parallel. \tag{42}$$

Expressions (40)–(42) are related by the normalization condition $\rho'^2 + \rho^2\theta'^2 + z'^2 = 1$. This is not immediately apparent. It can be shown by taking a derivative of equation (40) and squaring, together with the squares of equations (41), (42), as long as ρ does not vanish.

4. Bending

Let us consider an Hamiltonian that depends at most on the curvature,

$$H = \int \mathrm{d}s\, f(\kappa)$$

where f is any local function of its argument. We find that under an arbitrary deformation of the curve we have, with $f_\kappa = \partial f/\partial\kappa$,

$$\begin{aligned}\delta H &= \int \mathrm{d}s\{f_\kappa \delta_\perp \kappa - f\kappa\Psi_1\} + \int \mathrm{d}s(f\Psi_\parallel)' \\ &= \int \mathrm{d}s[f_\kappa'' + (\kappa^2 - \tau^2)f_\kappa - \kappa f]\Psi_1 + \int \mathrm{d}s[(2\tau f_\kappa)' - f_\kappa\tau']\Psi_2 \\ &\quad + \int \mathrm{d}s[f_\kappa\Psi_1' - f_\kappa'\Psi_1 + f\Psi_\parallel - 2\tau f_\kappa\Psi_2]'\end{aligned} \tag{43}$$

where we have used expression (11) for $\delta_\perp\kappa$. By comparison with equation (20), we immediately read off the Euler–Lagrange equations $\mathcal{E}_i = 0$, where

$$\mathcal{E}_1 = f_\kappa'' + (\kappa^2 - \tau^2)f_\kappa - \kappa f \tag{44}$$

$$\mathcal{E}_2 = (2\tau f_\kappa)' - \tau' f_\kappa. \tag{45}$$

We see that τ contributes to the 'driving force' for κ in the first equation. Integrating the second gives

$$f_\kappa^2\tau = \text{constant} \tag{46}$$

which determines τ as a function of κ. We show below that the constant appearing here is J as defined by equation (34). We can substitute into equation (44) for τ to obtain a second-order differential equation for κ. It is clear that the Euler–Lagrange equations (44) and (45) are integrable; τ is given as a function of κ, κ is determined as a quadrature.

The Noether charge $\mathcal{Q}$ is identified as the divergence appearing in equation (43),

$$\mathcal{Q} = f\Psi_\parallel + f_\kappa\Psi_1' - f_\kappa'\Psi_1 - 2\tau f_\kappa\Psi_2. \tag{47}$$

The conserved force $\boldsymbol{F}$ is obtained by specializing the deformation to a constant infinitesimal translation $\delta\boldsymbol{X} = \boldsymbol{e}$ in this expression. In the second term, we use the Frenet–Serret equation to obtain $\Psi_1' = \boldsymbol{e} \cdot \boldsymbol{n}_1' = \boldsymbol{e} \cdot (-\kappa\boldsymbol{t} + \tau\boldsymbol{n}_2)$. Equation (47) then gives

$$\boldsymbol{F} = (f_\kappa\kappa - f)\boldsymbol{t} + f_\kappa'\boldsymbol{n}_1 + \tau f_\kappa\boldsymbol{n}_2. \tag{48}$$

Note that the tension in the curve is identified as $-F_\parallel = f - f_\kappa\kappa$. It is not constant, in general, along the curve, and may also take negative values.

In a similar way, from the Noether charge corresponding to a rotation $\delta \boldsymbol{X} = \boldsymbol{\Omega} \times \boldsymbol{X}$ we obtain the conserved torque, $\boldsymbol{M}$. In general, only terms with derivatives of the Ψ_i contribute to the differential torque $\boldsymbol{T}$. In this case we have $\Psi_1' = (\boldsymbol{\Omega} \times \boldsymbol{X} \cdot \boldsymbol{n}_1)' = \boldsymbol{\Omega} \cdot (\boldsymbol{X} \times \boldsymbol{n}_1)' = \boldsymbol{\Omega} \cdot (\boldsymbol{t} \times \boldsymbol{n}_1 + \boldsymbol{X} \times \boldsymbol{n}_1')$. The second term contributes to the orbital part of $\boldsymbol{M}$, while from the first we find that the differential torque is given by

$$\boldsymbol{T} = -f_\kappa \boldsymbol{t} \times \boldsymbol{n}_1 = -f_\kappa \boldsymbol{n}_2. \tag{49}$$

Note that while the torque due to bending, $\boldsymbol{M}$, is not generally of the simple form $\boldsymbol{F} \times \boldsymbol{X}$ unless $f_\kappa = 0$, i.e. f is constant, neither is $\boldsymbol{T}$ of the most general form: there is no component along either $\boldsymbol{t}$ or $\boldsymbol{n}_1$ if only bending is penalized. This accords with our intuition: the axis of rotation due to the bending which rotates $\boldsymbol{t}$ towards $\boldsymbol{n}_1$ is along $\boldsymbol{n}_2$. For this model, the second Casimir of the Euclidean group, J, given by equation (34), is read off by dotting equations (48) and (49) as

$$FJ = -{f_\kappa}^2 \tau. \tag{50}$$

We thus identify the constant appearing in equation (46) as $-FJ$.

Substituting equation (50) for τ into the magnitude of the force determines κ as a quadrature, involving the two constants F and J:

$$F^2 = (f_\kappa')^2 + (f_\kappa \kappa - f)^2 + \frac{F^2 J^2}{(f_\kappa)^2}. \tag{51}$$

Suppose f_κ is not constant. The quadrature (51) describes the radial motion (f_κ) of a fictitious particle with a mass $= 2$, positive energy F^2 and angular momentum FJ, moving in the central potential, $V(\kappa) = (f_\kappa \kappa - f)^2$. We note that this potential is bounded from below. Equation (51) can be integrated to determine κ implicitly as a function of s:

$$s = \int \frac{\mathrm{d}f_\kappa}{\left[F^2 - V(f_\kappa) - F^2 J^2 / {f_\kappa}^2\right]^{1/2}}. \tag{52}$$

Once κ, and therefore τ via equation (50), is known, one can use expressions (40)–(42) to obtain by a further quadrature the trajectory in the adapted cylindrical coordinates $\{\rho, \theta, z\}$. In particular, we note that from equation (40) it follows that the radial coordinate ρ is determined pointwise by the curvature κ.

The physically most relevant model is the one quadratic in κ, subject to the constraint that the total length is constant [18]:

$$H = \int \mathrm{d}s (\kappa^2 + \mu) \tag{53}$$

where μ is a Lagrange multiplier enforcing the constraint. We have $f(\kappa) = \kappa^2 + \mu$, and equation (51) reduces to

$$F^2 = 4(\kappa')^2 + (\kappa^2 - \mu)^2 + \frac{F^2 J^2}{4\kappa^2} \tag{54}$$

where we have used $FJ = -4\kappa^2 \tau$. The potential becomes quartic for large κ. If J vanishes, $\tau = 0$, and the problem reduces to that of a planar elastica. There is one circular solution labelled by the winding number $n = \pm 1, \pm 2, \ldots$, corresponding to an n-fold covering of the circle, with $\kappa^2 = \mu$. μ is then fixed by L and n. The vector $\boldsymbol{F}$ vanishes on these solutions whereas $\boldsymbol{T}$ does not, pointing out of the plane. The doubly degenerate ground states are the once covered circles with $n = \pm 1$.

For $n = 0$, there are two oppositely oriented figure-eight configurations. There are two inflection points where $\kappa = 0$, connecting equal positive and negative curvature lobes. The integrability condition, equation (28), implies

$$\int \mathrm{d}s\, \kappa^2 = \mu L \tag{55}$$

where L is the total length of the loop—this fixes μ to be positive.

Consider next the model described by (see, e.g., [2])

$$H = \int \mathrm{d}s (\kappa - \kappa_0)^2 \tag{56}$$

where κ_0 is some positive constant, the 'spontaneous' curvature (note that we do not include a constant length constraint in this case). The absolute minimum, $H = 0$, is obtained when $\kappa = \kappa_0$, which corresponds to an n-fold circular loop of radius $R_0 = \kappa_0^{-1}$, with n arbitrary. The ground state is therefore infinitely degenerate. In this model we have,

$$F^2 = 4(\kappa')^2 + \left(\kappa^2 - \kappa_0^2\right)^2 + \frac{F^2 J^2}{4(\kappa - \kappa_0)^2}. \tag{57}$$

If $J = 0$, τ vanishes as well, and the potential possesses a single minimum at $\kappa = \kappa_0$. The only equilibria with $J = 0$ as before are the circles and figure eight. If $J \neq 0$, the potential is quite different from the one considered earlier: it now diverges at $\kappa = \kappa_0$. On either side of κ_0, it develops a local miminum. As before, the integrability condition can be used to exclude constant κ closed loops. Since $F_\| = \kappa^2 - \kappa_0^2$, the integrability condition then implies that $\kappa = \kappa_0$ if it is constant.

We end this section with a brief description of the model described by the scale-invariant bending energy with $f = \kappa$. In this case, equation (44) implies $\tau = 0$. We also have $\boldsymbol{F} = 0 = J$. Any plane loop extremizes this bending energy, which (for positive κ) coincides with the winding number of the loop on this plane. The minimum is realized on any convex loop.

5. Torsion

We now turn to Hamiltonians of the form

$$H = \int \mathrm{d}s\, f(\tau) \tag{58}$$

where f is an arbitrary local function of its argument. We will see that, like in the case $H = \int \mathrm{d}s\, f(\kappa)$ discussed in the previous section, such models are integrable by quadratures. We will require for the remainder of this section that neither κ or τ vanish.

We determine the normal variation of the free energy ($f_\tau = \partial f/\partial\tau$):

$$\begin{aligned} \delta_\perp H|_1 = &\int \mathrm{d}s \left[2\frac{\tau}{\kappa} f_\tau'' + \frac{\tau'}{\kappa} f_\tau' - 2\frac{\tau\kappa'}{\kappa^2} f_\tau' - f\kappa + 2\kappa\tau f_\tau\right]\Psi_1 \\ &+ \int \mathrm{d}s \left[2\frac{\tau}{\kappa} f_\tau \Psi_1' + \frac{\tau'}{\kappa} f_\tau \Psi_1 - 2\frac{\tau}{\kappa} f_\tau' \Psi_1\right]' \end{aligned} \tag{59}$$

$$\begin{aligned} \delta_\perp H|_2 = &\int \mathrm{d}s \left[-\left(\frac{f_\tau'}{\kappa}\right)'' + \frac{\tau^2 f_\tau'}{\kappa} - (\kappa f_\tau)'\right]\Psi_2 \\ &+ \int \mathrm{d}s \left[\frac{f_\tau}{\kappa}\Psi_2'' - \frac{f_\tau'}{\kappa}\Psi_2' + (\kappa^2 - \tau^2)\frac{f_\tau}{\kappa}\Psi_2 + \left(\frac{f_\tau'}{\kappa}\right)'\Psi_2\right]' \end{aligned} \tag{60}$$

where we have used equations (5), (14). The Euler–Lagrange derivatives $\mathcal{E}_i$ along the normal directions are identified as the coefficients of the Ψ^i discarding total derivatives

$$\mathcal{E}_1 = 2\tau \left(\frac{f_\tau'}{\kappa}\right)' + \frac{\tau'}{\kappa} f_\tau' - \kappa f + 2\kappa\tau f_\tau \tag{61}$$

$$\mathcal{E}_2 = -\left(\frac{f_\tau'}{\kappa}\right)'' + \frac{\tau^2 f_\tau'}{\kappa} - (\kappa f_\tau)'. \tag{62}$$

As expected, the Euler–Lagrange equations are of order 3 in derivatives of τ, and therefore of sixth order in derivatives of the embedding functions.

The Noether charge is given by the total derivatives in (59), (60),

$$\begin{aligned}\mathcal{Q} = f\Psi_\parallel + 2\frac{\tau}{\kappa} f_\tau \Psi_1' - \left[2\frac{\tau}{\kappa} f_\tau' - \frac{\tau'}{\kappa} f_\tau\right]\Psi_1 + \left(\frac{f_\tau}{\kappa}\right)\Psi_2'' - \frac{f_\tau'}{\kappa}\Psi_2' \\ + \left[\left(\frac{f_\tau'}{\kappa}\right)' + \left(\frac{f_\tau}{\kappa}\right)(\kappa^2 + \tau^2)\right]\Psi_2.\end{aligned} \tag{63}$$

This permits us to write the constant force as

$$\boldsymbol{F} = (\tau f_\tau - f)\boldsymbol{t} + \frac{\tau}{\kappa} f_\tau' \boldsymbol{n}_1 - \left[\left(\frac{f_\tau'}{\kappa}\right)' + \kappa f_\tau\right]\boldsymbol{n}_2. \tag{64}$$

We note that the structure of the tangential component is identical to the one of $\boldsymbol{F}$ for bending in equation (48), with $\tau \leftrightarrow \kappa$. Moreover, the second derivative of τ appearing in F_2 can be lowered to a first derivative by exploiting the Euler–Lagrange equation $\mathcal{E}_1 = 0$. Note that this was not necessary in first curvature models. We have

$$\left(\frac{f_\tau'}{\kappa}\right)' + \kappa f_\tau = \frac{1}{2\tau}\left[\kappa f - \tau'\left(\frac{f_\tau'}{\kappa}\right)\right] \tag{65}$$

so that we have the alternative expression:

$$\boldsymbol{F} = (\tau f_\tau - f)\boldsymbol{t} + \frac{\tau}{\kappa} f_\tau' \boldsymbol{n}_1 - \frac{1}{2\tau}\left[\kappa f - \tau' \frac{f_\tau'}{\kappa}\right]\boldsymbol{n}_2. \tag{66}$$

The differential torque $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = -f_\tau \boldsymbol{t} - \frac{f_\tau'}{\kappa}\boldsymbol{n}_1. \tag{67}$$

Note that in addition to the expected tangential component $-f_\tau \boldsymbol{t}$ due to the twist about the 'rod' axis (see, e.g., [21]), there is a contribution due to differential twist along $\boldsymbol{n}_1$. There is, however, no $\boldsymbol{n}_2$ component. The second Casimir, as defined by equation (34), takes the form

$$FJ = f_\tau(f - \tau f_\tau) - \tau\left(\frac{f_\tau'}{\kappa}\right)^2. \tag{68}$$

Remarkably, the solution by quadratures is possible exactly as in the first curvature models, as we show now for a special case which is sufficient for our purposes,

$$f = \tau^2/2 + \mu. \tag{69}$$

From equation (66) we obtain immediately

$$F^2 = \frac{\kappa^2}{4\tau^2}\left[\left(\frac{\tau'}{\kappa}\right)^2 - \frac{\tau^2}{2} - \mu\right]^2 + \frac{(\tau^2 - 2\mu)^2}{4} + \tau^2\left(\frac{\tau'}{\kappa}\right)^2. \tag{70}$$

Moreover, from equation (68), we obtain

$$FJ = \frac{1}{2}\tau(2\mu - \tau^2) - \tau\left(\frac{\tau'}{\kappa}\right)^2 \tag{71}$$

which permits us to eliminate τ'/k from equation (70). Doing this, we get

$$\tau^4(\tau^4 - 4\mu^2) + \kappa^2(\tau^3 + FJ)^2 + 4FJ\tau^5 + 4F^2\tau^4 = 0. \tag{72}$$

Thus, τ is determined in terms of J, F and κ as a root of an eighth-order polynomial. It is rather surprising that τ is determined pointwise, just as in the pure bending case, as some function of κ.

If we insist on adhering to the same mechanical analogue of a non-relativistic particle with radial κ exploited for bending, the potential appearing in the quadrature is going to be a mess. Fortunately, it is also possible to set up a quadrature for τ. We solve equation (72) for κ as a function of τ,

$$\kappa^2 = 4\tau^4 \frac{F^2 - \mu^2 + \tau FJ + \frac{1}{4}\tau^4}{(FJ + \tau^3)^2} \tag{73}$$

and substitute into equation (71). We obtain an equation of the form

$$\tau'^2 + V(\tau, F, J, \mu) = 0 \tag{74}$$

where the potential is given by

$$V(\tau, F, J, \mu) = 4\tau^3 \frac{\left(F^2 - \mu^2 + \tau FJ + \frac{1}{4}\tau^4\right)\left(\mu\tau - FJ - \frac{1}{2}\tau^3\right)}{(FJ + \tau^3)^2} \tag{75}$$

which again describes a non-relativistic particle (this time with position τ and zero 'energy') moving in a potential which is a ratio of polynomials. The analysis of the equilibrium configurations for this model is beyond the scope of this paper. We note that the potential tends asymptotically to $\tau^4/2$—a quartic once again. Note that the integrability condition

$$\int \mathrm{d}s(\tau^2 - 2\mu) = 0 \tag{76}$$

implies that μ must be positive on a closed loop.

6. Bending and torsion

Let us now consider models with a joint dependence on κ and τ, $f = f(\kappa, \tau)$. The integrability exhibited by the cases considered so far does not persist, in general, when f depends on both κ and τ. Unfortunately, this is also the case of interest in biophysics where, for example, models of the type $f = \alpha\kappa^2 + \beta\tau^2$ are studied.

The Euler–Lagrange equations take the form

$$\mathcal{E}_1 = 0 = f_\kappa'' + (\kappa^2 - \tau^2)f_\kappa - \kappa f + 2\tau\left(\frac{f_\tau'}{\kappa}\right)' + \tau'\frac{f_\tau'}{\kappa} + 2\kappa\tau\, f_\tau \tag{77}$$

$$\mathcal{E}_2 = 0 = 2(\tau f_\kappa)' - \tau' f_\kappa - \left(\frac{f_\tau'}{\kappa}\right)'' + \tau^2\frac{f_\tau'}{\kappa} - (\kappa f_\tau)'. \tag{78}$$

Note that one has to be careful to include only once the term $-\kappa f$ in adding equations (44) and (61) to obtain $\mathcal{E}_1$. The force and the differential torque are

$$\boldsymbol{F} = (f_\tau\tau + f_\kappa\kappa - f)\boldsymbol{t} + \frac{1}{\kappa}(\kappa f_\kappa' + \tau f_\tau')\boldsymbol{n}_1 - \left[\left(\frac{f_\tau'}{\kappa}\right)' + \kappa f_\tau - \tau f_\kappa\right]\boldsymbol{n}_2 \tag{79}$$

$$\boldsymbol{T} = -\left[f_\tau\boldsymbol{t} + \frac{f_\tau'}{\kappa}\boldsymbol{n}_1 + f_\kappa\boldsymbol{n}_2\right]. \tag{80}$$

The force is obtained by adding equations (48), (64), taking care to include the term $-f\boldsymbol{t}$ only once. The differential torque is given by the sum of equations (49) and (67). Note that $\boldsymbol{T}$ has components along all directions. It follows that the second Casimir is

$$FJ = -f_\tau(f_\tau\tau + f_\kappa\kappa - f) - \frac{f'_\tau}{\kappa^2}(f'_\kappa\kappa + f'_\tau\tau) + f_\kappa\left[\left(\frac{f'_\tau}{\kappa}\right)' + \kappa f_\tau - \tau f_\kappa\right]. \tag{81}$$

By comparing this expression for FJ with the corresponding one for the model depending only on τ, it is clear that the same strategy used in section 5 to produce a quadrature will not work. There are, however, two interesting mixed cases that are tractable by quadratures.

The first possibility is to consider the bending energy constrained to a fixed length and torsion (see [6] for a detailed analysis of this model). Thus, we consider the model defined by

$$f = \tfrac{1}{2}\kappa^2 + \alpha\tau + \mu. \tag{82}$$

The total torsion, $\mathcal{T} = \int \mathrm{d}s\,\tau$, is dimensionless. The fact that makes this model a minimal variation with respect to the pure bending models of section 4 is that it does not introduce derivatives of τ in the equilibrium conditions. The force and the differential torque are given by

$$\boldsymbol{F} = \tfrac{1}{2}(\kappa^2 - 2\mu)\boldsymbol{t} + \kappa'\boldsymbol{n}_1 + \kappa(\tau - \alpha)\boldsymbol{n}_2 \tag{83}$$

$$\boldsymbol{T} = -\alpha\boldsymbol{t} - \kappa\boldsymbol{n}_2. \tag{84}$$

We have then that the second Casimir is

$$FJ = \alpha\mu + \frac{\kappa^2}{2}(\alpha - 2\tau). \tag{85}$$

This invariant can be inverted for τ as

$$\tau = \frac{1}{\kappa^2}(\alpha\mu - FJ) + \frac{\alpha}{2}. \tag{86}$$

The corresponding quadrature then takes the form

$$\kappa'^2 + \frac{1}{4}(\kappa^2 - 2\mu)^2 + \frac{1}{4\kappa^2}\left(\alpha\mu - FJ + \frac{1}{2}\kappa^2\right)^2 = F^2. \tag{87}$$

Note that the torsion constraint does not affect the integrability condition (28) on $F_\|$, although it affects the form of the coordinates ρ, θ as defined in section 3.4.

The second possibility is given by adding a term linear in κ to the model (69), so that

$$f = \tfrac{1}{2}\tau^2 + \alpha\kappa + \mu. \tag{88}$$

The two Casimirs now take the form

$$F^2 = \frac{\kappa^2}{4\tau^2}\left[\left(\frac{\tau'}{\kappa}\right)^2 - \frac{\tau^2}{2} - \mu + \alpha\frac{\tau^2}{\kappa}\right]^2 + \frac{(\tau^2 - 2\mu)^2}{4} + \tau^2\left(\frac{\tau'}{\kappa}\right)^2 \tag{89}$$

$$FJ = \frac{1}{2}\tau(2\mu - \tau^2) - \left(\tau + \frac{\alpha\kappa}{2\tau}\right)\left(\frac{\tau'}{\kappa}\right)^2 + \frac{\alpha\kappa}{2\tau}\left(\mu + \frac{\tau^2}{2} - \frac{\alpha\tau^2}{\kappa}\right). \tag{90}$$

Note that the latter is considerably more complicated than in the pure torsion case, as is given by equation (71). It is possible to use FJ to eliminate τ'/κ in F^2, so that τ is determined by κ pointwise. However, the resulting expression is quite messy.

Although it is clear that the general case will not be reducible to a quadrature, the use of the Casimirs of the Euclidean group allows for significant simplifications over a direct approach at the level of the equilibrium conditions. We illustrate this fact with an example: let us look at the model

$$f = \tfrac{1}{2}(\kappa^2 + \tau^2). \tag{91}$$

This is known as total curvature [23], and it is a natural function of curvature and torsion, in the sense that $\boldsymbol{n}_1' \cdot \boldsymbol{n}_1' = \kappa^2 + \tau^2$. It also appears in [24] as a conserved Hamiltonian. From equations (77), (78), we read off the equilibrium conditions

$$\mathcal{E}_1 = 2\tau \left(\frac{\tau'}{\kappa}\right)' + \frac{\tau'^2}{\kappa} + \kappa'' + \frac{\kappa}{2}(\kappa^2 - \tau^2) = 0 \tag{92}$$

$$\mathcal{E}_2 = -\left(\frac{\tau'}{\kappa}\right)'' + \frac{\tau'\tau^2}{\kappa} + \tau\kappa' = 0. \tag{93}$$

For the force and differential torque, from equations (79), (80), we obtain

$$\boldsymbol{F} = \frac{1}{2}(\kappa^2 + \tau^2)\boldsymbol{t} + \frac{1}{2\kappa}(\kappa^2 + \tau^2)'\boldsymbol{n}_1 - \left(\frac{\tau'}{\kappa}\right)' \boldsymbol{n}_2 \tag{94}$$

$$\boldsymbol{T} = -\left(\tau\boldsymbol{t} + \frac{\tau'}{\kappa}\boldsymbol{n}_1 + \kappa\boldsymbol{n}_2\right). \tag{95}$$

It follows that the Casimir invariants take the form

$$\begin{aligned} F^2 &= \left[\left(\frac{\tau'}{\kappa}\right)'\right]^2 + \frac{1}{4\kappa^2}[(\kappa^2 + \tau^2)']^2 + \frac{1}{4}(\kappa^2 + \tau^2)^2 \\ FJ &= \kappa\left(\frac{\tau'}{\kappa}\right)' - \frac{\tau'}{2\kappa^2}(\kappa^2 + \tau^2)' - \frac{\tau}{2}(\kappa^2 + \tau^2). \end{aligned} \tag{96}$$

We can eliminate the second derivative in F^2 using the definition of J to provide a condition of the form

$$\mathcal{F}(\kappa, \kappa', \tau, \tau', F, J) = 0 \tag{97}$$

which can be considered as the energy condition for the motion of a fictitious particle in two dimensions.

7. A note on perturbations

We have, so far, focused on exact methods in dealing with the Euler–Lagrange equations for the elastic models we consider. We digress briefly in this section to comment on the perturbative analysis of these equations—a more expanded treatment of this material will be presented elsewhere [25]. Although outside the main focus of this paper, we find it instructive to include here an example of a complementary approach which, through approximations, allows a complete treatment, from the energy functional to the actual embedding that minimizes it. Apart from the obvious benefit to intuition, we obtain a non-trivial check of many of our formulae by computing the force $\boldsymbol{F}$ and the Casimir FJ and verifying that they are constant. We do this in the particular case

$$f = \kappa^2 + \tau^2 + \mu^2 \tag{98}$$

i.e., when the total curvature is penalized, with constrained length.

Using equations (77), (78), the equilibrium conditions can be satisfied, with f as above, by constant (non-zero) curvature and torsion, $\kappa = \kappa_0$, $\tau = \tau_0$, provided that

$$\kappa_0^2 + \tau_0^2 = \mu^2 \tag{99}$$

(note that $\mathcal{E}_2$ is identically zero for constant curvature and torsion). The resulting space curve is a circular helix. Perturbations would give to the axis of this helix a small curvature and torsion, while changing, in general, κ_0 and τ_0 as well. We then take κ and τ to be power series in a small parameter ϵ (related to the above 'macroscopic' curvature and torsion of the axis of the helix) and read off the resulting Euler–Lagrange equations order-by-order in ϵ. The zeroth-order result is equation (99) above, while to order ϵ we get

$$\kappa_1'' + 2\tau_0\tau_1'' + \kappa_1 + \tau_0\tau_1 = 0 \qquad \tau_1''' - \tau_0\kappa_1' - \tau_0^2\tau_1' = 0 \tag{100}$$

where

$$\kappa(s) = \kappa_0 + \epsilon\kappa_1(s) + \mathcal{O}(\epsilon^2) \qquad \tau(s) = \tau_0 + \epsilon\tau_1(s) + \mathcal{O}(\epsilon^2) \tag{101}$$

and we have set $\kappa_0 = 1$. The solutions to (100) involve constant terms, sines and cosines, as well as terms proportional to s and s^2. To reduce the number of parameters (five initial conditions as well as μ), we choose to eliminate the terms in s and s^2, resulting in the constraints

$$\kappa_1'(0) = (2\mu^2 - 1)\tau_0^{-1}\tau_1'(0) \qquad \tau_1''(0) = \tau_0(\kappa_1(0) + \tau_0\tau_1(0)). \tag{102}$$

This amounts to a restriction to periodic solutions, with period equal to that of the unperturbed helix. The following abbreviations will be useful in this section

$$\alpha_1 \equiv \mu^{-2}(\tau_0\kappa_1(0) + \alpha_2\tau_1(0)) \qquad \alpha_2 \equiv 2\mu^2 - 1 \qquad \alpha_3 \equiv \mu^{-2}(\kappa_1(0) + \tau_0\tau_1(0)).$$

We may furthermore set $\tau_1'(0) = 0$ by a suitable shift in s. The solutions then become

$$\kappa_1(s) = -\alpha_1\tau_0 + \alpha_2\alpha_3\cos(\mu s) \qquad \tau_1(s) = \alpha_1 - \alpha_3\tau_0\cos(\mu s). \tag{103}$$

We now determine the corresponding embedding, using the Weierstrass representation for the curve (we follow the conventions in [16]). We first solve the differential equation

$$\Phi'(s) = Q(s)\Phi(s) \tag{104}$$

where $\Phi(s)$ is the $SU(2)$ matrix describing the rotation of the Frenet–Serret frame and

$$Q(s) = -\tau(s)e_0 - \kappa(s)e_2 \tag{105}$$

with $e_0 = -\frac{\mathrm{i}}{2}\sigma^3$, $e_1 = -\frac{\mathrm{i}}{2}\sigma^1$, $e_2 = -\frac{\mathrm{i}}{2}\sigma^2$ and σ^i the Pauli matrices. The embedding is then given by

$$\tilde{x}(s) + \mathrm{i}\tilde{y}(s) = \int_0^s 2\alpha\bar{\beta}\,\mathrm{d}s' \qquad \tilde{z}(s) = \int_0^s (\alpha\bar{\alpha} - \beta\bar{\beta})\,\mathrm{d}s' \tag{106}$$

where $\alpha = \Phi_{11}$ and $\beta = \Phi_{12}$. This gives us the helix with its tangent vector, at $s = 0$, along $\hat{z}$. To get instead its axis, at $s = 0$, along $\hat{z}$, we rotate around the x-axis by an angle η, with $\tan\eta = \tau_0/\kappa_0$. Denoting the resulting embedding by (x, y, z), we find

$$\begin{aligned} x(s) &= -\mu^{-2}\cos(\mu s) + \epsilon\left(\mu^{-2}\tau_0\alpha_1(\cos(\mu s) - 1) - \tfrac{1}{4}\alpha_3(\cos(2\mu s) - 1) + \tfrac{1}{2}\tau_0^2\alpha_3 s^2\right) \\ y(s) &= -\mu^{-2}\sin(\mu s) + \epsilon\left(\mu^{-2}\tau_0\alpha_1\sin(\mu s) - \tfrac{1}{4}\alpha_3\sin(2\mu s) + \left(\tfrac{1}{2}\mu\alpha_3 - \mu^{-1}\tau_0\alpha_1\right)s\right) \\ z(s) &= \mu^{-1}\tau_0 s + \epsilon\left(\mu^{-2}(\alpha_1 + \tau_0\alpha_3)\sin(\mu s) + \mu^{-1}\tau_0\alpha_3 s\cos(\mu s) + \mu^{-1}\alpha_1 s\right). \end{aligned} \tag{107}$$

We note that the axis of the helix is bent in the x–z plane due to the s^2 term in $x(s)$, while the term linear in s in $y(s)$ gives it torsion as well. As a check of our general formulae in the previous sections, as well as of our explicit calculations in this section, we compute now $\boldsymbol{t}$, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ and then the force $\boldsymbol{F}$ from (79). We find

$$\boldsymbol{F} = -2\epsilon\mu^3\alpha_3\boldsymbol{y} \tag{108}$$

showing that the (indeed constant) force only appears as a result of the perturbation, an exclusive feature of this particular model. Finally, equation (81) gives that FJ is zero to this order in ϵ, $FJ = 0 + \mathcal{O}(\epsilon^2)$.

A final remark is due concerning the validity of the above first-order solution. When moving on to second order, the solutions for κ_2, τ_2, will again involve periodic as well as non-periodic terms. Eliminating the latter fixes some of the parameters that are arbitrary in the first-order results. More generally, requiring periodicity at order *k*, restricts the solutions found to orders less than *k*, a feature that can be traced to the nonlinearity of the Euler–Lagrange equations.

8. Filament model recursion scheme

In this section, we discuss briefly the filament model recursion scheme, and its relationship with the Noether currents for curves. It is described in great detail by Langer in [16], and our brief discussion is contained in his work. We are only changing the point of view by putting Hamiltonians to the forefront, rather than curve motions.

As we mentioned briefly at the end of section 2, consider spatial vector fields $\boldsymbol{Y}$ which locally preserve arclength, then the filament model recursion scheme is defined by

$$\boldsymbol{t} \times \boldsymbol{Y}_{(n)} = \boldsymbol{Y}'_{(n-1)} \qquad \text{with} \quad \boldsymbol{Y}_{(0)} = -\boldsymbol{t}. \tag{109}$$

The first few terms in this hierarchy are

$$\begin{aligned} \boldsymbol{Y}_{(1)} &= \kappa \boldsymbol{n}_2 \\ \boldsymbol{Y}_{(2)} &= \frac{\kappa^2}{2}\boldsymbol{t} + \kappa' \boldsymbol{n}_1 + \kappa\tau \boldsymbol{n}_2 \\ \boldsymbol{Y}_{(3)} &= \kappa^2\tau \boldsymbol{t} + (2\tau\kappa' + \kappa\tau')\boldsymbol{n}_1 + \left(\kappa\tau^2 - \kappa'' - \frac{\kappa^3}{2}\right)\boldsymbol{n}_2. \end{aligned}$$

These vector fields have remarkable properties. First, we recognize that $\boldsymbol{Y}_{(1)}$, known as the filament model, is the differential torque $\boldsymbol{T}$ for the model $f = \frac{1}{2}\kappa^2$, and $\boldsymbol{Y}_{(2)}$ is both the force $\boldsymbol{F}$ for the same model and also the differential torque for the model $f = \kappa^2\tau$. Moreover, the integral of the tangential component of the *n*th order vector field gives the corresponding conserved Hamiltonian.

Now, is it possible to set up alternative recursion schemes? From the point of view of the Hamiltonians, one can start with some $H = \int \mathrm{d}s\, f(\kappa, \tau)$, compute its differential torque $\boldsymbol{T}$ set as $\boldsymbol{T} = \boldsymbol{Z}_{(1)}$. Then from equation (33) it follows that the associated force is $\boldsymbol{F} = \boldsymbol{Z}_{(2)}$. Now from the equilibrium conditions in the form (25), we have $\mathcal{E}^i \boldsymbol{n}_i = \boldsymbol{t} \times \boldsymbol{Z}_{(3)}$, so that $\boldsymbol{Z}_{(3)} = Z_{(3)\|}\boldsymbol{t} + \mathcal{E}_2\boldsymbol{n}_1 - \mathcal{E}_1\boldsymbol{n}_2$. However, to satisfy the condition that the vector be arclength preserving, we need to satisfy $Z'_\| - \kappa\mathcal{E}_2 = 0$, and this 'perfect derivative phenomenon' appears to happen only in the filament model recursion scheme. For example, for the model quadratic in τ, $\kappa\mathcal{E}_2$ is not a total derivative.

Acknowledgments

RC was supported by CONACyT grant 32187-E. CC and JG were supported by CONACyT grant 32307-E and DGAPA-UNAM grant IN119792.

References

[1] Doi M and Edwards S F 1988 *The Theory of Polymer Dynamics* (Oxford: Clarendon)
[2] Goldstein R E and Langer S A 1995 *Phys. Rev. Lett.* **75** 1094
[3] Bouchiat C and Mezard M 1998 *Phys. Rev. Lett.* **80** 1156
[4] Marko J F and Siggia E D 1994 *Macromolecules* **27** 981
Marko J F and Siggia E D 1995 *Macromolecules* **28** 8759

[5] Giaquinta M and Hildebrandt S 1996 *Calculus of Variations I* (Berlin: Springer)
[6] Ivey T A and Singer D A 1999 *Proc. Lond. Math. Soc.* **79** 429
[7] Willmore T J 2000 Curves *Handbook of Differential Geometry* 997 (Amsterdam: Elsevier)
[8] Kamien R D 2002 *Preprint* cond-mat/0203127
[9] Hasimoto H 1972 *J. Fluid Mech.* **51** 477
[10] Lamb G L Jr 1976 *Phys. Rev. Lett.* **37** 235
Lamb G L Jr 1976 *Phys. Rev. Lett.* **37** 723 (erratum)
[11] Goldstein R E and Petrich D N 1992 *Phys. Rev. Lett.* **69** 555
[12] Lakshmanan M 1977 *Phys. Lett.* A **61** 53
[13] Balakrishnan R 1982 *Phys. Lett.* A **92** 243
[14] Goldstein R E and Petrich D N 1991 *Phys. Rev. Lett.* **67** 3203
[15] Langer J and Perline R 1994 *J. Math. Phys.* **35** 1732
[16] Langer J 1999 *NY J. Math.* **5** 25
[17] Spivak M 1979 *A Comprehensive Introduction to Differential Geometry* vol 4, 2nd edn (Houston: Publish or Perish)
[18] Langer J and Singer D 1984 *J. Diff. Geom.* **20** 1
[19] Bryant R and Griffiths P 1986 *Am. J. Math.* **108** 525
[20] Arreaga G, Capovilla R, Chryssomalakos C and Guven J 2002 *Phys. Rev.* E **65** 031801
[21] Landau L D and Lifshitz E M 1970 *Theory of Elasticity* 2nd edn (Oxford: Pergamon)
[22] Langer J and Singer D 1984 *J. Lond. Math. Soc.* **30** 512
[23] Struik D J 1988 *Lectures on Classical Differential Geometry* 2nd edn (New York: Dover)
[24] Murugesh S and Balakrishnan R 2001 *Phys. Lett.* A **290** 81
[25] Capovilla R, Chryssomalakos C and Guven J 2002 in preparation